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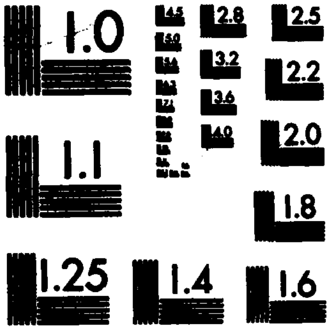

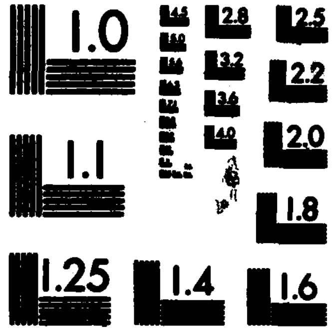

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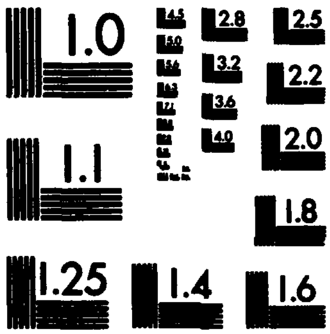

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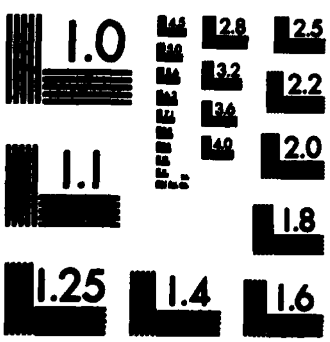

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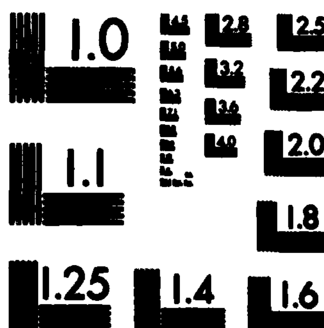

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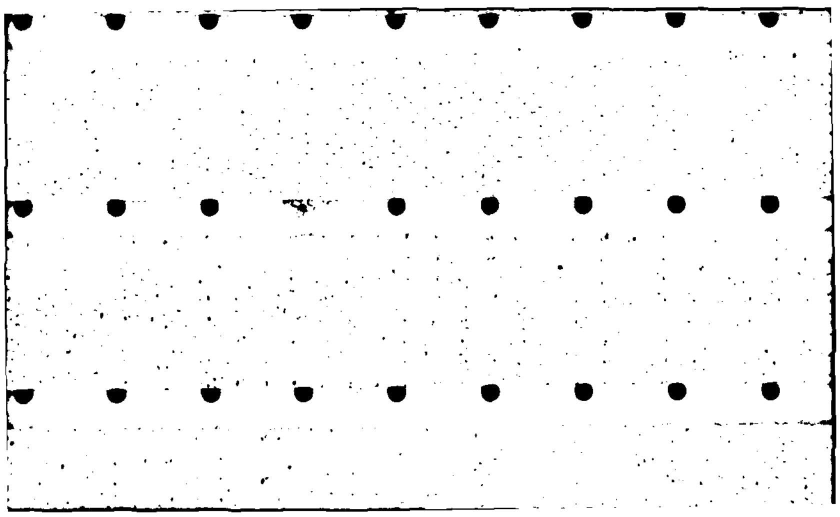

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ERROR MODELS FOR STABLE HYBRID ADAPTIVE SYSTEMS

PART II

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Error Models for Stable Hybrid Adaptive Systems

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~~Abstract:~~ The paper presents a unified approach to the design of stable adaptive algorithms for hybrid, discrete and continuous systems which have two time scales. Hybrid error models are first analyzed in detail using two distinct approaches and the results are extended to discrete systems in which the parameters are adjusted at rates slower than those at which the systems operate. The algorithms developed when applied to the adaptive control problem are shown to result in global stability. Simulation results are included and reveal the dependence of the convergence rates of the various algorithms on the two time-scales.

Introduction: In the past few years several continuous [1,2] and discrete [3,4] adaptive algorithms have been developed for the stable identification and control of linear time-invariant plants with unknown parameters. In continuous adaptive systems the plant operates in continuous time and the controller parameters are adjusted continuously. Similarly, in discrete adaptive systems, the various signals as well as the control parameter vector are defined at discrete instants of time. However, practical adaptive systems, for a variety of reasons, may not be either purely discrete or purely continuous. For example, recent advances in microprocessors and related digital computer technology favor the use of discrete controls even for continuous systems. Such systems are now referred to as hybrid control systems. Further, even in purely discrete systems, the computational effort involved quite often necessitates the adjustment of the adaptive control parameters at rates significantly slower than that at which the system operates.

Both hybrid as well as discrete adaptive systems of the latter type can be considered as systems with two time-scales. Recently, continuous algorithms of an integral type have been suggested in the literature [5] which can also be interpreted as belonging to this class. This paper proposes a unified approach to the analysis and synthesis of such systems.

Gawthrop [6] was the first to introduce the concept of hybrid self-tuning controllers which are partly realized in continuous time and partly in discrete time. In [7] Elliott presented a method for the indirect control of hybrid adaptive systems and more recently Cristi and Monopoli [8] attempted the direct hybrid control of model reference adaptive systems. The central question in both cases is the global stability of the overall system which can be best analyzed by considering the behavior of the corresponding error model. The main contribution of this paper is the detailed analysis of several error models using two distinct approaches. The insights obtained by this analysis, as well as the specific results derived provide valuable tools for the synthesis of stable adaptive controllers for general two time-scale systems and for hybrid adaptive systems in particular.

In section 2, using a Lyapunov approach, different adaptive laws are developed for the various error models which assure the boundedness of the parameter error vector. Conditions under which the output and parameter error vectors tend to zero asymptotically, are also derived and are particularly useful in the identification problem. The main result of this section, which is most relevant to the control problem, relates the rate of growth of the output error to that of the norm of the input vector. While this result is common to all the error models, the method of proof used in each case is found to be significantly different.

Applications, extensions and modifications of the adaptive algorithms are treated briefly in section 3. The principal steps in the proof of global stability of a hybrid adaptive control system are first outlined in section 3a. The same

algorithms when suitably modified are also shown in section 3b to be applicable to both discrete and continuous systems with two time-scales. Finally, in section 3c it is shown that the use of time-varying adaptive gain matrices (as in the well known recursive least-squares approach) in the algorithms does not affect the stability arguments of the earlier sections.

Section 4 contains detailed simulation results of different error models. The speed and accuracy of the various algorithms are compared for different values of the period T during which the control parameters are constant. Approach 1 is found to be better for small values of T while approach 2 is significantly better for larger values. A convex combination of the two schemes, which can also be demonstrated to be stable, is suggested as a viable alternative for most adaptive systems.

2. Error Models:

The dynamical systems discussed in this section are continuous time systems in which $t \in \mathbb{R}^+$, the set of all positive real numbers. Let $u: \mathbb{R}^+ \to \mathbb{R}^m$ and $e_1: \mathbb{R}^+ \to \mathbb{R}$ be piecewise continuous functions referred to as the input and output functions of the error models respectively. Let $\{t_k\}$ be a monotonically increasing unbounded sequence in $\mathbb{R}^+$ with $0 < T_{min} \leq T_k \leq T_{max} < \infty$ for $k \in \mathbb{N}$, where T_k is defined as $T_k \overset{\Delta}{=} t_{k+1} - t_k$. When $t_k = kT$ (or $T_k = T$) we shall call T the sampling period. Let $\phi: \mathbb{R}^+ \to \mathbb{R}^m$ be a piecewise constant function, referred to as the parameter error vector and assume values

$$\phi(t) = \phi_k \qquad t \in [t_k, t_{k+1})$$

where ϕ_k is a constant vector. The error models described in this section relate the output error e_1, the input vector u and parameter error vector ϕ in terms of algebraic or differential equations.

a) Error Model 1:

The first error model is described by the equation

$$\phi_k^T u(t) = e_1(t) \qquad t \in [t_k, t_{k+1}) \quad k \in N \tag{1}$$

It is assumed that ϕ_0 (and hence ϕ_k) is unknown, the values $u(t)$ and $e_1(t)$ can be observed at every instant t and $\Delta\phi_k \triangleq \phi_{k+1} - \phi_k$ can be adjusted at $t = t_{k+1}$. The objective is to determine an adaptive law for choosing the sequence $\{\Delta\phi_k\}$ using all available input-output data so that

$$\lim_{t\to\infty} e_1(t) = 0.$$

Approach 1:

Consider the Lyapunov function candidate

$$V(k) = 1/2\ \phi_k^T \phi_k \tag{2}$$

Then

$$\begin{aligned}\Delta V(k) &\triangleq V(k+1) - V(k)\\ &= [\phi_k + \frac{\Delta\phi_k}{2}]^T \Delta\phi_k\end{aligned} \tag{3}$$

Choosing the adaptive law

$$\Delta\phi_k = -\frac{1}{T_k}\int_{t_k}^{t_{k+1}} \frac{e_1(\tau)u(\tau)}{1 + u^T(\tau)u(\tau)}\, d\tau \tag{4}$$

yields

$$\Delta V(k) = -\frac{1}{2}\phi_k^T [2I - R_k] R_k \phi_k \tag{5}$$

where R_k is the symmetric positive semi-definite matrix

$$R_k = \frac{1}{T_k}\int_{t_k}^{t_{k+1}} \frac{u(\tau)u^T(\tau)}{1 + u^T(\tau)u(\tau)}\, d\tau \tag{6}$$

with all its eigenvalues within the unit circle. Since $[2I-R_k] > \beta I$ for some

constant $\beta > 0$,

$$\Delta V(k) < -\frac{1}{2}\beta\phi_k^T R_k \phi_k \leq 0 \tag{7}$$

Hence $V(k)$ is a Lyapunov function and assures the boundedness of $\|\phi_k\|$ if $\|\phi_0\|$ is bounded. Further, since $\{\Delta V(k)\}$ is a non-negative sequence with $\sum_{k=0}^{\infty} \Delta V(k) < \infty$ it follows that $\Delta V(k) \to 0$ as $k \to \infty$ or alternately from (7) $\phi_k^T R_k \phi_k \to 0$ as $k \to \infty$. This can also be expressed as

$$\frac{1}{T_k}\int_{t_k}^{t_{k+1}} \frac{e_1^2(\tau)}{1 + u^T(\tau)u(\tau)} d\tau \to 0 \text{ as } k \to \infty \tag{8}$$

Case (i): If u is uniformly bounded in $\mathbb{R}^+$ it follows from (1) that e_1 is also uniformly bounded. Since $\sum_{k=0}^{\infty} \Delta V(k)$ is bounded we have from (8) that $e_1 \in L^2$. Further if $\dot{u}$ is also bounded, then $\lim_{t\to\infty} e_1(t) = 0$. Hence for a uniformly bounded u with a uniformly bounded derivative, e_1 tends to zero and $\Delta\phi_k \to 0$ as $k \to \infty$.

Case (ii): *If in addition to being uniformly bounded* u is "sufficiently rich" [9] over an interval T_{min}, so that the matrix R_k is positive definite for all $k \in N$, then $\Delta V(k) < 0$. Hence the parameter error vector tends to zero as $k \to \infty$.

Case (iii): A more interesting case arises when u may grow at most exponentially and is relevant to the control problem (see comments at the end of this section). Since condition (8) is independent of the assumption of the boundedness of u, it follows that $\frac{e_1}{\sqrt{1 + u^T u}} \in L^2$. If the input satisfies the condition $\|\dot{u}\| \leq M_1 \|u\| + M_2$ for some constants M_1 and M_2, and hence grows at most exponentially, $\frac{e_1(t)}{\sqrt{1 + u^T(t)u(t)}}$ has a bounded derivative and the integrand in (8) tends to zero as $k \to \infty$. Hence

$$\phi_k^T u(t) = e_1(t) = o[\sup_{t \geq \tau} \|u(\tau)\|] \tag{9}$$

or e_1 grows slowly [1] compared to the norm of the input vector u. Equation (9)

plays a central role in the proof of stability of the hybrid adaptive control problem and is discussed in section 3a .

Approach 2: In this approach the error equation (1) is modified to the equivalent form

$$\phi_k^T e_1(t)u(t) = e_1^2(t) \qquad t \in [t_k, t_{k+1}) \quad k \in N \tag{10}$$

by multiplying the two sides by $e_1(t)$. Integrating over the interval $[t_k, t_{k+1})$ yields:

$$\phi_k^T \int_{t_k}^{t_{k+1}} e_1(\tau)u(\tau)d\tau = \int_{t_k}^{t_{k+1}} e_1^2(\tau)d\tau$$

or equivalently the discrete error model

$$\phi_k^T \omega(k) = \epsilon(k) \tag{11}$$

where

$$\int_{t_k}^{t_{k+1}} e_1(\tau)u(\tau)d\tau \triangleq \omega(k) \quad \text{and} \quad \int_{t_k}^{t_{k+1}} e_1^2(\tau)d\tau \triangleq \epsilon(k).$$

For the error model (11) the adaptive law for updating ϕ_k can be written by inspection [3] as

$$\Delta\phi_k = \frac{-\epsilon(k)\omega(k)}{1 + \omega(k)^T\omega(k)} \quad . \tag{12}$$

From well known results in discrete adaptive control [3,4] it also follows that

(i) $\|\phi_k\|$ is bounded if $\|\phi_0\|$ is bounded

(ii) $\Delta\phi_k \to 0$ as $k \to \infty$

and (iii) $|\epsilon(k)| = o[\sup_{k \geq \nu}\|\omega(\nu)\|]$ or $\epsilon(k)$ grows more slowly than $\omega(k)$.

Case (i): As in approach 1, if u is uniformly bounded in $\mathbb{R}^+$, e_1 and hence $\epsilon(k)$ and $\omega(k)$ are uniformly bounded for all $t \in \mathbb{R}^+$ and $k \in N$ respectively. If $\|\dot{u}\|$ is also bounded $\lim_{t\to\infty} e_1(t) = 0$.

Case (ii): If in addition to being uniformly bounded u is sufficiently rich over an interval T_{min}, $\int_{t_k}^{t_{k+1}} u(\tau)u^T(\tau)d\tau = P_k$ is positive definite for all $k \in N$. If $V(k) = \frac{1}{2}\phi_k^T\phi_k$, the adaptive law (12) results in $\Delta V(k) = -\frac{[\phi_k^T P_k \phi_k]^2}{1 + \phi_k^T P_k^2 \phi_k} < 0$ for all $\phi_k \neq 0$ and $k \in N$. Hence $\phi_k \to 0$ as $k \to \infty$.

Case (iii): As in approach 1, the case of greatest interest arises when $u \in L_e^\infty$, $u \notin L^\infty$ and grows at most exponentially. The condition $|\epsilon(k)| = o[\sup_{k \geq \nu} \|\omega(\nu)\|]$ can be equivalently represented as

$$\int_{t_k}^{t_{k+1}} e_1^2(\tau)d\tau = o[\sup_{k \geq \nu} \| \int_{t_\nu}^{t_{\nu+1}} e_1(\tau)u(\tau)d\tau \|]. \tag{13}$$

Equation (13) implies that a sequence $\{\beta(k)\}$ exists with $\beta(k) \geq 0 \quad \forall k \in N$ and $\beta(k) \to 0$ as $k \to \infty$, such that

$$\int_{t_k}^{t_{k+1}} |e_1(\tau)|[|e_1(\tau)| - \beta(k)\| u(\tau)\|]d\tau = 0 \qquad \forall k \in N \tag{14}$$

Since $e_1(\tau) = \phi_k^T u(\tau)$, $\tau \in [t_k, t_{k+1})$ and ϕ_k is bounded (14) can also be written as

$$\int_{t_k}^{t_{k+1}} |e_1(\tau)| \| u(\tau)\| [\nu(k,\tau) - \beta(k)]d\tau = 0 \qquad \forall k \in N \tag{15}$$

where $\nu(k,\tau) \triangleq \|\phi_k\| \, |\cos \omega_k(\tau)|$ and $\omega_k(\tau)$ is the angle between the vectors ϕ_k and $u_k(\tau)$ for $\tau \in [t_k, t_{k+1})$. Since $\beta(k) \to 0$ as $k \to \infty$, $\frac{d}{dt}\nu(k,t)$ is bounded and $\nu(k,\tau) \geq 0$ for $\tau \in [t_k, t_{k+1})$ it follows that $\nu(k,\tau) \to 0$ as $k \to \infty$ which in turn implies that

$$e_1(t) = \phi_k^T u(t) = o[\sup_{t \geq \tau} \| u(\tau)\|]$$

which is the same as (9) derived using approach 1.

b) Error Model 2:

The second error model is described by the error differential equation

$$\dot{e} = Ae + b\phi^T u \tag{16}$$

where $e(t), b \in R^n$, $A \in R^{n\times n}$ and is stable, (A,b) is controllable, $\phi(t), u(t) \in R^m$ and $\phi(t) = \phi_k$, $t \in [t_k, t_{k+1})$, $k \in N$ where ϕ_k is a constant vector. In this case the parameter error vector is to be adjusted using the input u and the state error vector e. Since A is a stable matrix, a symmetric positive definite matrix $P = P^T > 0$ exists such that $A^TP + PA = -Q < 0$.

Approach 1: Using this approach error model (16) is first modified, as in the continuous case [1], using a feedback term $-\gamma u^T ub^T Pe$ to the form

$$\dot{e} = Ae + b[\phi^T u - \gamma u^T ub^T Pe] \tag{17}$$

The corresponding adaptive law is the average gradient obtained in the continuous case over one period and is given by

$$\Delta\phi_k = -\frac{1}{T_{max}}\int_{t_k}^{t_{k+1}} u(\tau)b^T Pe(\tau)\,d\tau \tag{18}$$

Defining the *Lyapunov function candidate* as

$$V(k) = \frac{1}{T_{max}} e^T(t_k)Pe(t_k) + \phi_k^T\phi_k \tag{19}$$

$$\Delta V(k) = -\frac{1}{T_{max}}\int_{t_k}^{t_{k+1}} e^T(\tau)Qe(\tau)\,d\tau - \frac{2\gamma}{T_{max}}\int_{t_k}^{t_{k+1}}[e^T(\tau)Pb]^2u^T(\tau)u(\tau)\,d\tau$$

$$+\frac{1}{T_{max}^2}\left\|\int_{t_k}^{t_{k+1}} e^T(\tau)Pbu(\tau)\,d\tau\right\|^2 \tag{20}$$

Since for any vector $x(t) \in R^n$ $\quad 1/T\int_{t-T}^{t}\|x(\tau)\|^2 d\tau \geq [1/T\|\int_{t-T}^{t} x(\tau)\,d\tau\|]^2$ the second term on the right-hand-side of equation (20) dominates the third term for all $\gamma \geq \frac{T_{max}}{2T_{min}}$. Hence

$$\Delta V(k) \leq 0 \qquad \forall\, k \in N \;\text{ and }\; \gamma \geq \frac{T_{max}}{2T_{min}} \tag{21}$$

This implies that $\|\phi_k\|$ and $\|e(t_k)\|$ are bounded if $\|\phi_0\|$ and $\|e(t_0)\|$ are bounded and $\Delta V(k) \to 0$ as $k \to \infty$ i.e.

$$\int_{t_k}^{t_{k+1}} e^T(\tau)Qe(\tau)\,d\tau \to 0\;;\; \int_{t_k}^{t_{k+1}}[e^T(\tau)Pb]^2u^T(\tau)u(\tau)\,d\tau \to 0 \qquad \text{as } k \to \infty \tag{22}$$

Again, since $\sum_{k=0}^{\infty} \Delta V(k)$ is bounded $e,[e^T Pb]u \in L^2$.

Case (i): If u is uniformly bounded, it follows from equation (17) that $\dot{e}$ is also uniformly bounded. Since $e \in L^2$ this results in $\lim_{t\to\infty} e(t) = 0$.

Case (ii): As in the earlier cases with error model 1, when $u(\cdot)$ is sufficiently rich over a period T_{min}, $V(k)$ is monotonically decreasing and hence $\phi_k \to 0$ as $k \to \infty$.

Case (iii): A somewhat more involved argument is needed when $u \in L_e^{\infty}, u \notin L^{\infty}$ and grows at most exponentially to relate the growth rates of u and $\phi_k^T u$. The existence of the discrete Lyapunov function assures the boundedness of e(t) at the discrete instants $t = t_k$. Since e can grow at most exponentially by equation (17) and since the interval $[t_k, t_{k+1})$ is uniformly bounded, e(t) is bounded for all $t \in R^+$.

Now, the same arguments as those used in the continuous case [1] can be used to demonstrate that $\| W_M(s)\phi^T u \| = o[\sup_{t\geq\tau} \| u(\tau)\|]$ where $W_M(s) \triangleq [sI-A]^{-1}b$. In equation (17) $[\phi^T u - \gamma u^T u b^T Pe]$ is the input to an exponentially stable system. Since $e^T Pbu \in L^2$ the component of e resulting from $e^T Pbu^T u$ must grow at a slower rate than $\|u\|$ i.e. $o[\sup_{t\geq\tau} \| u(\tau)\|]$. If this is unbounded, then the response due to $\phi^T u$ should also be $o[\sup_{t\geq\tau} \| u(\tau)\|]$, since we have already demonstrated that e(t) is uniformly bounded for $t \in R^+$.

Approach 2: In this case the error model is described by

$$\dot{e} = Ae + b\phi_k^T u \qquad t \in [t_k, t_{k+1}) \tag{23}$$

and does not contain the feedback term as in (17). Multiplying both sides of (23) by $e^T P$ and equating the integrals over an interval $[t_k, t_{k+1})$

$$e^T(t)Pe(t)\Big|_{t_k}^{t_{k+1}} + \int_{t_k}^{t_{k+1}} e^T(\tau)Qe(\tau)\,d\tau = 2\int_{t_k}^{t_{k+1}} e^T(\tau)Pbu^T(\tau)\phi_k\,d\tau \tag{24}$$

or

$$\phi_k^T w(k) = \varepsilon(k) \tag{25}$$

where l.h.s. of (24) is $\varepsilon(k)$ and $2\int_{t_k}^{t_{k+1}} e^T(\tau)Pbu(\tau)\,d\tau = w(k)$. Once again the adaptive law may be written by inspection as

$$\Delta\phi_k = \frac{-\varepsilon(k)w(k)}{1 + w^T(k)w(k)} \tag{26}$$

and yields $$|\varepsilon(k)| = o[\sup_{k\geq\nu} \| w(\nu)\|] \tag{27}$$

Case (i): If u is uniformly bounded, from equation (23) we have $e(t)$ and $\dot{e}(t)$ uniformly bounded. Hence $w(k)$ is uniformly bounded and $\varepsilon(k) \to 0$ as $k \to \infty$. Since

$$\varepsilon(k) = e^T(t)Pe(t)\Big|_{t_k}^{t_{k+1}} + \int_{t_k}^{t_{k+1}} e^T(\tau)Qe(\tau)\,d\tau \tag{28}$$

this implies that $\lim_{k\to\infty} e^T(t)Pe(t)\Big|_{t_k}^{t_{k+1}} = 0$ and the integral in (28) tends to zero as $k \to \infty$. Since $\dot{e}(t)$ is uniformly bounded $\lim_{t\to\infty} e(t) = 0$.

Case (ii): If u is sufficiently rich over any interval of length T_{min}, $\phi_k^T u(t) \to 0$ implies $\phi_k \to 0$ as $k \to \infty$.

Case (iii): For the case where u grows at most exponentially we prove that $\phi_k^T u(t) = o[\sup_{t\geq\tau} u(\tau)]$ by contradiction. Defining $e^T Pb \triangleq e_1$ (see error model 3) (27) may be written as:

$$\left| e^T(t)Pe(t)\Big|_{t_k}^{t_{k+1}} + \int_{t_k}^{t_{k+1}} e^T(\tau)Qe(\tau)\,d\tau \right| = \beta(k) \left\| \int_{t_k}^{t_{k+1}} e_1(\tau)u(\tau)\,d\tau \right\| \tag{29}$$

where $\beta(k) \geq 0$, $k \in N$ and $\beta(k) \to 0$ as $k \to \infty$. Let $\phi^T(t)u(t) \neq o[\sup_{t\geq\tau}\|u(\tau)\|]$. Then we can denote $\| u(t)\| = O[\sup_{t\geq\tau}|\phi^T(\tau)u(\tau)|]$. In such a case, from (23) $\| e(t)\|$ and $\| u(t)\|$ grow at the same rate and equation (29) can be modified to

$$\left| e^T(t_{k+1})Pe(t_{k+1}) - e^T(t_k)Pe(t_k) + \int_{t_k}^{t_{k+1}} e^T(\tau)Qe(\tau)\,d\tau \right| = \beta_1(k) \int_{t_k}^{t_{k+1}} e_1^2(\tau)\,d\tau \tag{30}$$

where $\beta_1(k) \to 0$ as $k \to \infty$. From (30) it follows that the sequence $\{e^T(t_{k+1})Pe(t_{k+1}) - e^T(t_k)Pe(t_k)\} \to 0$ as $k \to \infty$ and $\int_{t_k}^{t_{k+1}} e^T(\tau)Qe(\tau)\,d\tau \to 0$ as $k \to \infty$. Since $\| e(t)\|$ can grow at most exponentially the latter implies that $e(t) \to 0$ as

$t \to \infty$ which contradicts the assumption that $\|e(t)\|$ and $\|u(t)\|$ grow without bound.

Error Model 3:

The third hybrid error model is merely a special case of the second error model but is important in view of its practical applications - e.g. adaptive control of a plant with relative degree 1. It is described by the scalar differential equation

$$\dot{e}_1(t) = -\alpha e_1(t) + [\phi_k^T u(t) - \gamma e_1(t) u^T(t) u(t)]$$

if approach 1 is used and by the equation

$$\dot{e}_1(t) = -\alpha e_1(t) + \phi_k^T u(t) \tag{31}$$

if approach (2) is used. The corresponding adaptive laws can be specialized from (18) and (26) as:

$$\Delta\phi_k = -\frac{1}{T_{max}} \int_{t_k}^{t_{k+1}} e_1(\tau) u(\tau) d\tau \qquad \text{(approach 1)}$$

and $$\Delta\phi_k = \frac{-\varepsilon(k) w(k)}{1 + w^T(k) w(k)} \qquad \text{(approach 2)} \tag{32}$$

where

$$\varepsilon(k) \triangleq e_1^2(t)\Big|_{t_k}^{t_{k+1}} + \alpha \int_{t_k}^{t_{k+1}} e_1^2(\tau) d\tau \tag{33}$$

and $$w(k) \triangleq 2 \int_{t_k}^{t_{k+1}} e_1(\tau) u(\tau) d\tau \tag{34}$$

In both cases it can be shown that

$$\left| W_M(s) \phi_k^T u(t) \right| = o[\sup_{t \geq \tau} \|u(\tau)\|] \tag{35}$$

where $W_M(s) = \frac{1}{s + \alpha}$.

Comments:

(i) The detailed analysis presented in this section is based on the conviction of the authors that efficient design methods for adaptive systems can arise only from a deeper understanding of the behavior of corresponding error

models. For each model considered three specific cases have been discussed. The first two assume that the input u to the error model is uniformly bounded. The results are particularly relevant to the identification problem where the plant to be identified is assumed to be stable and the input to the plant is uniformly bounded. When the hybrid adaptive algorithms described in this section are used to identify such a plant, the output errors will tend to zero and the parameters will tend to the true values if the input is sufficiently rich. The error model used and the specific algorithm chosen depend upon the parametrization of the plant and the sampling period T.

(ii) The main result of this section is relevant to the control problem as well as identification problems where the vector u cannot be guaranteed to be uniformly bounded a priori (e.g. the parallel model). When any one of the adaptive laws is used in such cases $\phi_k^T u$ (and $W_M(s)\phi_k^T u$ in error models 2 and 3) is shown to grow asymptotically at a rate slower than that at which $\sup_{t\geq\tau} \|u(\tau)\|$ grows. As shown in the next section this is central to the proof of global stability of the hybrid adaptive control problem.

(iii) The two approaches used to develop the adaptive algorithms in the three error models are conceptually different. In the first approach, the discrete Lyapunov function is a quadratic form in the parameter error vector (model 1) or both parameter and output vectors (models 2 and 3). The direction in which $\Delta\phi$ is adjusted is the average gradient of $e_1^2(t)$ with respect to ϕ over an interval $[t_k, t_{k+1})$. In contrast to this the second approach attempts to minimize the integral of $e_1^2(t)$ over an interval so that $\Delta\phi$ is the gradient of this performance index. As seen from the simulation results in section 4, the two adaptive algorithms lead to quite different responses of the overall system. Approach 1 is found to be more effective for smaller values of T_k while approach 2

is significantly better when T_k is large. Using the same Lyapunov function it can be shown that a convex combination of the two adaptive laws also assures the boundedness of the parameter errors. Hence such a combination of the two may be used in practice to realize their combined advantages.

(iv) As in purely discrete and continuous systems the adaptive laws in hybrid systems are also chosen to assure that the parameter vector ϕ varies slowly as $k \to \infty$. In the first error model this is accomplished using a factor $\frac{1}{1 + u^T u}$ in the adaptive law (4) in the first approach and the factor $\frac{1}{1 + w^T(k)w(k)}$ using the second (12). The feedback term in (17) serves the same purpose in error models 2 and 3 when the first approach is used and is omitted in the second approach where a quadratic factor $\frac{1}{1 + w^T(k)w(k)}$ is used (26).

3. Application, Extension and Refinement:

The concepts and techniques developed in section 2 find wide application in adaptive systems where practical considerations demand a hybrid approach. The most obvious of such applications is the design of stable hybrid adaptive controllers and is considered in section 3a. The same concepts can also be extended to discrete systems where data is collected at a faster rate than that at which the parameters are adjusted. This is briefly outlined in 3b. For the sake of completeness it is also shown in section 3c that algorithms of an integral type [5] suggested for continuous systems can be considered as natural generalizations of the algorithms developed for hybrid and discrete systems. Finally, well known methods for adjusting the adaptive gain matrix in a time-varying fashion e.g. recursive least-squares, to improve the speed of response can be readily extended to the above cases and are discussed in 3c.

a) Stable Hybrid Adaptive Control:

A hybrid adaptive control system is one in which a plant with unknown parameters which operates in continuous time is controlled by adjusting a controller parameter vector only at discrete instants. In [10] the adaptive law (4) was used to demonstrate the global stability of the overall system. Similar arguments can be used to show that all the other adaptive algorithms in section 2, when properly applied also result in global stability. For single-input single-output systems the adaptive algorithms generated using the first and third error models are suitable; the second error model has applications in multivariable control. In this section we merely outline the principal steps involved in the proof of global stability.

A linear time-invariant plant with unknown parameters is to be controlled adaptively. The input and output of the plant are $u(\cdot)$ and $y_p(\cdot)$ respectively and the plant transfer function $W_p(s)$ is rational with known order and relative degree and all zeros in the left half plane. A reference model has a stable rational transfer function $W_M(s)$ and has the same relative degree as the plant. The input $r(\cdot)$ of the model is piecewise continuous and uniformly bounded. The output of the model is $y_m(\cdot)$ and the aim of adaptive control is to generate an input $u(\cdot)$ to the plant using a differentiator free controller so that

$$\lim_{t\to\infty} |e_1(t)| = \lim_{t\to\infty} |y_p(t) - y_m(t)| = 0.$$

The solution to this problem for both the continuous [1,2] and discrete cases [3,4] is well known. In the former using the input $u(\cdot)$ and output $y_p(\cdot)$ the controller generates a vector of sensitivity functions $w(\cdot)$ so that the input $u(\cdot)$ to the plant can be expressed as

$$\theta^T(t)w(t) = u(t) \qquad \forall\ t \geq t_0 \tag{36}$$

where in general θ, w: $\mathbb{R}^+ \to \mathbb{R}^{2n+1}$. It is known that a constant vector θ^* exists such that when $\theta(t) \equiv \theta^*$ the transfer function of the plant together with the

controller is $W_M(s)$ and $\lim_{t\to\infty} e_1(t) = 0$. The aim of the adaptation procedure is to adjust $\dot{\theta}(t)$ (and hence $\theta(t)$) continuously so that this asymptotic behavior is realized.

In the hybrid adaptive control problem, the controller structure is identical to that used in the continuous case but the parameter vector $\theta(t)$ is adjusted only at discrete instants so that

$$\theta(t) = \theta_k \qquad t \in [t_k, t_{k+1}) \quad k \in N \tag{37}$$

where $\theta_k \in R^{2n+1}$ and is a constant vector. If $\theta(t) - \theta^* \triangleq \phi(t)$ the parameter error vector $\phi(t)$ is also piecewise constant and $\phi(t) = \phi_k$ for $t \in [t_k, t_{k+1})$ where $\theta_k - \theta^* = \phi_k$. The error equation for the control problem can be written as

$$W_M(s)\phi^T(t)w(t) = e_1(t) \tag{38}$$

Special cases exist for the control problem, *as for example when* $W_M(s)$ is a strictly positive real transfer function. We consider below, only the general case when $W_M(s)$ has a relative degree $n^* \geq 2$.

To generate an adaptive law for adjusting $\theta(t)$, an auxiliary signal $y_a(\cdot)$ is added to $e_1(\cdot)$ where

$$y_a(t) = [\theta^T(t)W_M(s)I - W_M(s)\theta^T(t)]w(t) \tag{39}$$

so that
$$\phi^T(t)\zeta(t) = e_1(t) + y_a(t) \triangleq \varepsilon_1(t) \tag{40}$$

where $\varepsilon_1(t)$ is referred to as the augmented error. Using the observed values of $\varepsilon_1(\cdot)$ and $\zeta(\cdot)$ the control parameter vector $\theta(\cdot)$ (and hence $\phi(\cdot)$) is adjusted and it is then shown that this results in a bounded plant output.

In section 2, the principal result of the analysis was that the adaptive laws (4) and (12) when applied to equation (40) assure that

$$\varepsilon_1(t) = \phi^T(t)\zeta(t) = o[\sup_{t\geq\tau} \|\zeta(\tau)\|] \tag{41}$$

Further, since $\zeta(t) = W_M(s)Iw(t)$ and $\Delta\phi(t) \to 0$ as $t \to \infty$ it can be shown [10] that

$$\begin{aligned}\phi^T(t)w(t) &= \phi^T(t)\zeta(t) + o[\sup_{t\geq\tau} \|w(\tau)\|] \\ &= o[\sup_{t\geq\tau} \|w(\tau)\|]\end{aligned} \tag{42}$$

From equation (38) it follows that the output error $e_1(\cdot)$ and hence the plant output $y_p(\cdot)$ can grow at most at a rate slower than $\sup_{t\geq\tau} \|w(\tau)\|$ which in turn assures their boundedness.

b) Extension to Systems with Two Time Scales:

The hybrid error models described in section 2 can be considered as systems which operate on two time scales -- a time scale associated with the continuous time functions and a second with the discrete parameters. Such situations also arise frequently in purely discrete systems where the inputs and outputs are observed at a certain rate but the control parameters are adjusted at a slower rate. The concepts established in section 2 are shown in this section to carry over directly to such systems also. Further, it is interesting to note that algorithms recently suggested for adjusting control parameters in continuous time systems [5] can be interpreted as the continuous counterparts of such discrete time systems operating on two time scales. Hence the methods suggested in section 2 can be considered to provide a unified approach to two time scale problems in discrete, continuous and hybrid systems.

(i) Discrete Time Models:

The first error model corresponding to error model (1) can be described by the equation

$$\phi_k^T u_\ell = e_\ell \qquad k, \ell \in N, \qquad \ell \in [kT,(k+1)T] \tag{43}$$

where ϕ_k is a constant vector in the interval $[kT,(k+1)T]$. Using approach 1 it can be shown that if the adaptive law

$$\phi_{(k+1)} - \phi_k = \Delta\phi_k = -\frac{1}{T}\sum_{i=kT}^{(k+1)T-1} \frac{e_i u_i}{1 + u_i^T u_i} \triangleq -R_k \phi_k \tag{44}$$

is used, $V(k) = \frac{1}{2}\phi_k^T \phi_k$ is a Lyapunov function. This, in turn, implies $\|\phi_k\|$ is bounded if $\|\phi_0\|$ is bounded and

$$\Delta V(k) \triangleq V(k+1) - V(k) = -\phi_k^T [I - \frac{1}{2} R_k] R_k \phi_k \leq 0 \tag{45}$$

which yields

$$\lim_{i\to\infty} \frac{e_i}{\sqrt{1 + u_i^T u_i}} = 0 \qquad i \in N \tag{46}$$

Equation (46) assures global stability of the adaptive control problem when the adaptive law (44) is used to adjust the control parameters in a discrete adaptive system.

Using the second approach, new variables $\varepsilon(k)$ and $\zeta(k)$ are defined as

$$\sum_{\ell=kT}^{(k+1)T-1} e_\ell^2 \triangleq \varepsilon(k) \quad ; \quad \sum_{\ell=kT}^{(k+1)T-1} e_\ell u_\ell = \zeta(k) \tag{47}$$

and the error model (43) can be expressed as

$$\phi_k^T \zeta(k) = \varepsilon(k) \qquad\qquad k \in N \tag{48}$$

The corresponding adaptive law is given by

$$\Delta\phi_k = \frac{-\varepsilon(k)\zeta(k)}{1 + \zeta^T(k)\zeta(k)} \tag{49}$$

and assures that $\lim_{i\to\infty} \frac{e_i}{\sqrt{1 + u_i^T u_i}} = 0$.

The other adaptive laws for the discrete error models corresponding to hybrid error models 2 and 3 in section 2 can be derived in a similar fashion.

(ii) Continuous Time Models:

In the continuous time error model

$$\phi^T(t)u(t) = e_1(t) \qquad t \in R^+ \tag{50}$$

it is well known that the adaptive law

$$\dot{\phi}(t) = \frac{-e_1(t)u(t)}{1 + u^T(t)u(t)} \tag{51}$$

results in a bounded parameter error vector. Recently other continuous adaptive laws have been suggested [5] which utilize past input-output data in adjusting adaptive parameters. We shall refer to such adaptive algorithms as integral algorithms in contrast to the point algorithm (51). By a proper definition of the error model such algorithms can be shown to be generalizations of the hybrid and discrete algorithms developed in sections 2 and 3b.

Let the unknown parameter error vector at time t be $\phi(t)$ and let the output $e_1(\cdot,\cdot):R^+ \times R^+ \to R$ be defined by

$$\phi^T(t)u(\tau) = e_1(t,\tau) \qquad \tau \leq t \qquad \tau,\ t \in R^+. \tag{52}$$

The adaptive law

$$\dot{\phi}(t) = -\frac{1}{T}\int_{t-T}^{t} \frac{u(\tau)e_1(t,\tau)}{1 + u^T(\tau)u(\tau)}\, d\tau \tag{53}$$

which uses input-output data over the interval $[t-T,t)$ is a generalization of (44) for the discrete case. However, unlike the discrete algorithm, (53) poses two major difficulties in implementation. The first involves the storage of the values of the function $u(\cdot)$ over the interval of integration. The second and significantly greater problem is caused by the fact that $e_1(t,\tau)$ cannot be measured directly for use in the adaptive law and cannot be computed from equation (53) since $\phi(t)$ is unknown.

The above problem can be circumvented by noting that error models of the type (52) arise in adaptive situations (as described in section 3a) where a parameter vector $\theta(t)$ is adjusted and has to evolve to a desired but unknown constant vector θ^* i.e. $\phi(t) = \theta(t) - \theta^*$.

Hence the error model (52) becomes

$$[\theta(t) - \theta^*]^T u(\tau) = e_1(t,\tau) \tag{54}$$

or

$$\theta^T(t)u(\tau) - y_m(\tau) = e_1(t,\tau) \tag{55}$$

where $y_m(\tau)$ is the signal produced by the model and can be measured. Hence in such cases the adaptive law can be implemented as

$$\dot{\phi}(t) = \dot{\theta}(t) = -\frac{1}{T}\int_{t-T}^{t} \left\{ \frac{u(\tau)u^T(\tau)\theta(t) - u(\tau)y_m(\tau)}{1 + u^T(\tau)u(\tau)} \right\} d\tau \tag{56}$$

As mentioned earlier, the implementation of (56) is rendered difficult by the fact that the values of $u(\cdot)$ have to be stored over a window of length T. To overcome this problem the length of the interval T is increased to t so that the entire past data is used but a weighting factor $e^{-q(t-\tau)}$ is included to assure the convergence of the integral. Such an exponentially weighted adaptive algorithm has the form

$$\dot{\phi}(t) = \dot{\theta}(t) = \int_0^t e^{-q(t-\tau)} \left\{ \frac{u(\tau)u^T(\tau)\theta(t) - u(\tau)y_m(\tau)}{1 + u^T(\tau)u(\tau)} \right\} d\tau \tag{57}$$

which can be conveniently realized by the matrix differential equations

$$\begin{aligned} \dot{\theta}(t) &= -R(t)\theta(t) - r(t) & \theta(t_0) &= 0 \\ \dot{R}(t) &= -qR(t) + \frac{u(t)u^T(t)}{1 + u^T(t)u(t)} & R(t_0) &= 0 \\ \dot{r}(t) &= -qr(t) + \frac{u(\tau)y_m(\tau)}{1 + u^T(\tau)u(\tau)} & r(t_0) &= 0\,. \end{aligned} \tag{58}$$

The adaptive law (58) is precisely the one suggested in [5].

In conclusion, the approach developed in section 2 is seen to unify discrete, continuous and hybrid adaptive algorithms with two time scales.

c) Adaptive Gain:

In the discussions in the preceding sections, adaptive gains were not included in the adaptive laws to focus attention on the principal results. Experience with complex adaptive systems has however shown that the speed of convergence of the algorithms depends critically on the choice of the adaptive gains. In particular a time-varying gain matrix obtained from least-squares considerations is found to be generally acceptable for most applications. In this section it is briefly shown that similar time-varying gain matrices can also be included in the hybrid adaptive laws. The details are included only for the first error model. Similar arguments carry over to the other error models also.

If the first error model is described by

$$\phi_k^T u(t) = e_1(t) \qquad t \in [t_k, t_{k+1}) \tag{59}$$

let

$$R_k \triangleq \frac{1}{T_k}\int_{t_k}^{t_{k+1}} \frac{u(\tau)u^T(\tau)}{1 + u^T(\tau)u(\tau)}\, d\tau \ .$$

The adaptive gain matrix Γ_k is defined by

$$\Gamma_{k+1}^{-1} = \Gamma_k^{-1} + R_k \qquad \Gamma_0 = I \tag{60}$$

and the adaptive law is given by

$$\Delta\phi_k = -\frac{\Gamma_k}{T_k}\int_{t_k}^{t_{k+1}} \frac{e_1(\tau)u(\tau)}{1 + u^T(\tau)u(\tau)}\, d\tau \tag{61}$$

For the system described by equations (59), (60) and (61) it can be shown that

$$V(k) = \frac{1}{2}\,\phi_k^T \Gamma_k^{-1} \phi_k \tag{62}$$

is a Lyapunov function resulting in

$$\Delta V(k) = -\frac{1}{2T_k}\int_{t_k}^{t_{k+1}} \frac{e_1^2(\tau)d\tau}{1+u^T(\tau)u(\tau)} - \frac{1}{T_k^2}\left[\int_{t_k}^{t_{k+1}} \frac{e_1(\tau)u^T(\tau)}{1+u^T(\tau)u(\tau)}d\tau\right][I-\Gamma_k R_k]\Gamma_k$$

$$\times\left[\int_{t_k}^{t_{k+1}} \frac{e_1(\tau)u(\tau)}{1+u^T(\tau)u(\tau)}d\tau\right] \tag{63}$$

$$\leq 0 \quad \text{if } I - \Gamma_k R_k > 0 \qquad \forall\, k \in N$$

From (63) it follows that $\|\phi_k\|$ is bounded if $\|\phi_0\|$ is bounded, $\Delta\phi_k \to 0$ as $k \to \infty$ and $\int_{t_k}^{t_{k+1}} \frac{e_1^2(\tau)}{1+u^T(\tau)u(\tau)}d\tau \to 0$ as $k \to \infty$. If $u(\cdot)$ grows at most exponentially, the last result implies that $e_1(t) = o[\sup_{t\geq\tau}\|u(\tau)\|]$.

4. Simulations:

The error models described in section 2 and their applications described in section 3 have been simulated extensively on the digital computer. We include in this section four typical examples which compare the effectiveness of the two adaptive approaches proposed in section 2. In all cases the parameters are adjusted periodically with a period T, so that $t_k = kT$ $(k \in N)$. The main interest in these simulations is on the effect of T on the speed and accuracy of the responses.

Example 1: The first hybrid error model, described by the equation (1) was simulated when $u(t),\phi(t) \in R^2$ and the input vector u is defined by

$$u_1(t) = \sin(.75t) \qquad u_2(t) = \sin(2.6t)$$

Figures 1a-1d show the evolution of the output error $e_1(t)$ and the parameter error vector $\phi(t)$ when approaches 1 and 2 are used. In Figures 1a and 1b T = 0.5 while T = 5.0 in Figures 1c and 1d. Approach 1 results in rapid convergence of $e_1(t)$ and $\phi(t)$ for T = 0.5. In contrast to this the convergence is very slow when approach 2 is used. This may be attributed to the fact that $w(k) \triangleq \int_{t_k}^{t_{k+1}} e_1(\tau)u(\tau)d\tau$ does not

vary adequately over an interval when the period T is small. As T is increased the response using the first approach deteriorates while that using the second approach improves markedly. With $T = 5$ it is seen that ϕ decreases to a value close to zero at the end of one period.

Example 2: Figures 2a-2d show the evolution of $e_1(t)$ and $\phi(t)$ when the same experiments as in example 1 were performed on error model 3. The basic features of the responses using the two approaches remain the same indicating that the approach rather than the specific error model chosen governs the behavior of the transient response.

Example 3: In this example, all the signals of interest are discrete, though input and output are defined for all $k \in N$ and the parameter error vector is adjusted periodically with period $T \in N$. In the second order system simulated

$$u_1(k) = \sin(.05k) \qquad u_2(k) = \sin(.25k)$$

The adaptive law used in this case to adjust $\Delta\phi$ had the form

$$\Delta\phi(k) = \delta\Delta\phi_1(k) + (1-\delta)\Delta\phi_2(k) \qquad 0 \leq \delta \leq 1$$

where $\Delta\phi_1(k)$ and $\Delta\phi_2(k)$ are the adaptive laws given by the two approaches. As might be expected $\delta \approx 1$ for small values of T and $\delta \approx 0$ for large values of T are found to be satisfactory as seen in Figure 3a-3d.

Example 4: The behavior of the error model when the input grows at most exponentially has been stressed throughout the paper. In particular the main result of section 2 was that in such a case the output error would grow at most at a slower rate as compared to the input. The experiments in this example were performed to verify this result. If u(t) has the form

$$u_1(t) = e^{.4t}\sin(.75t)$$
$$u_2(t) = e^{.4t}\sin(2.6t)$$

it is seen from Figure 4a-4b $\phi_k \to 0$ as $k \to \infty$ so that $e_1(t) \to 0$ as $t \to \infty$. In this case u grows in an unbounded fashion but is sufficiently rich. However, when only the first component $u_1(t)$ of $u(t)$ grows exponentially, ϕ_k does not tend to zero but asymptotically approaches a constant value orthogonal to the vector [1,0]. This can be seen in Figure 4c-4d.

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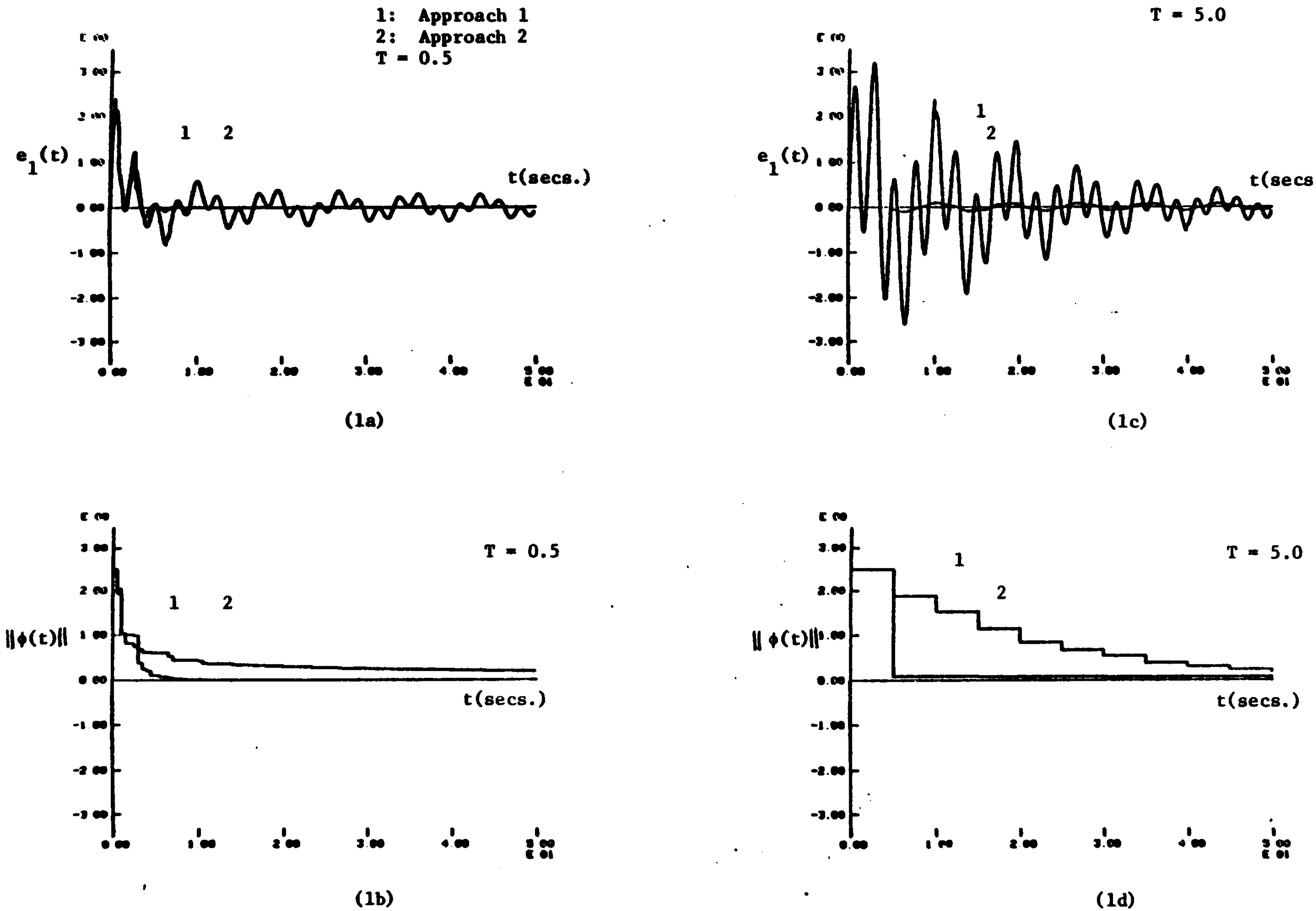

Figure 1: Evolution of $e_1(t)$ and $\phi(t)$ - Error Model 1

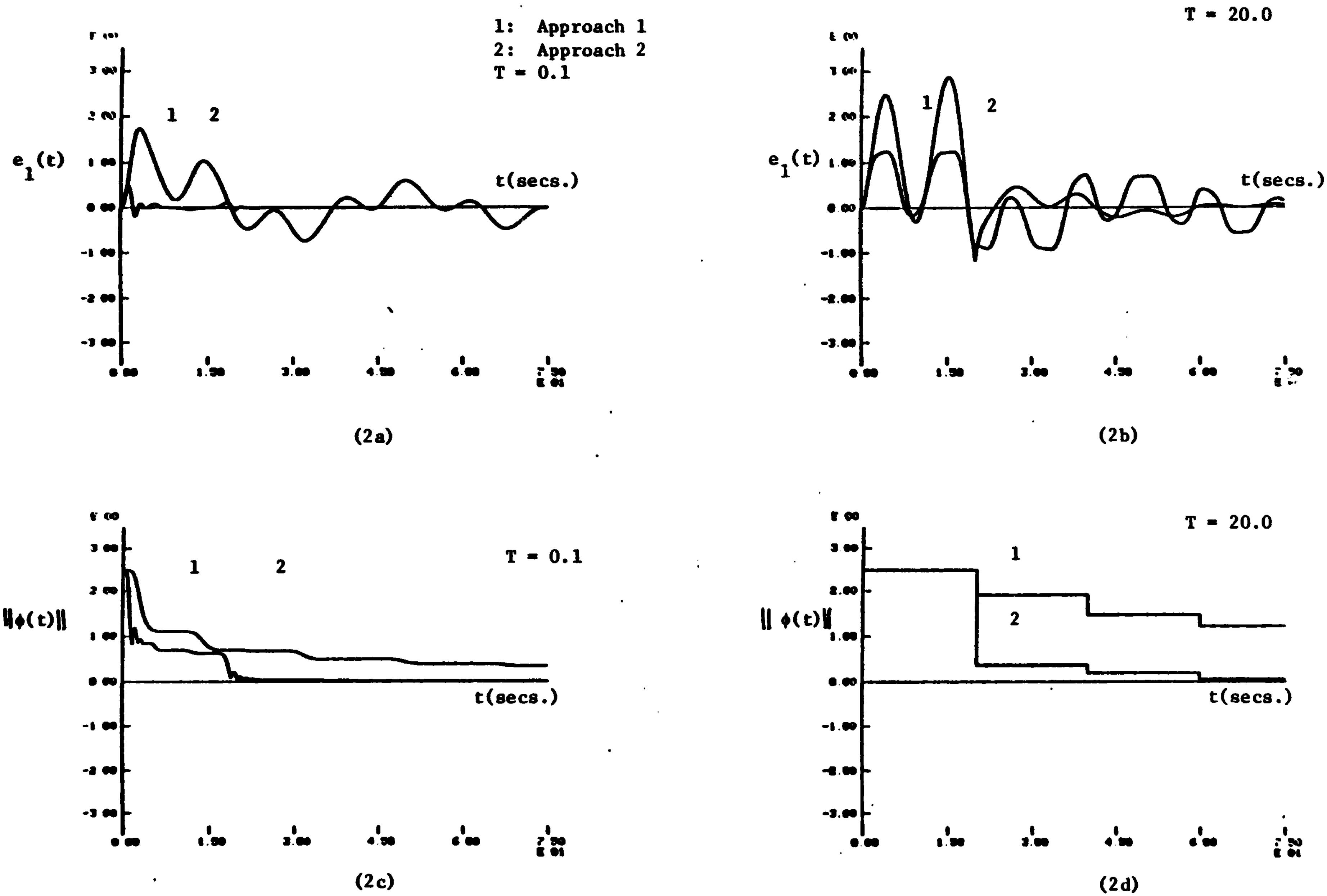

Figure 2: Evolution of $e_1(t)$ and $\phi(t)$ - Error Model 3

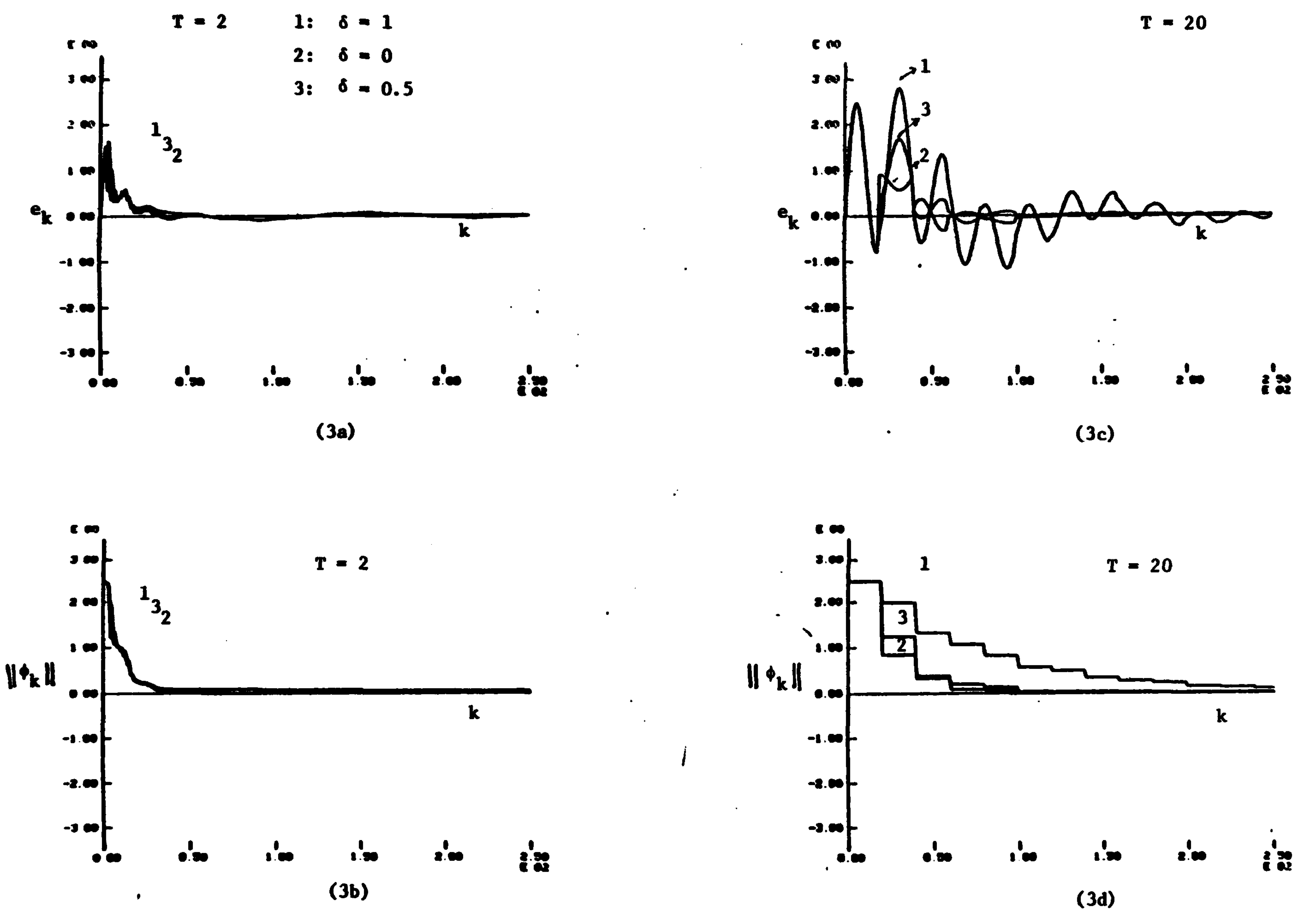

Figure 3: Discrete error model with two time scales.

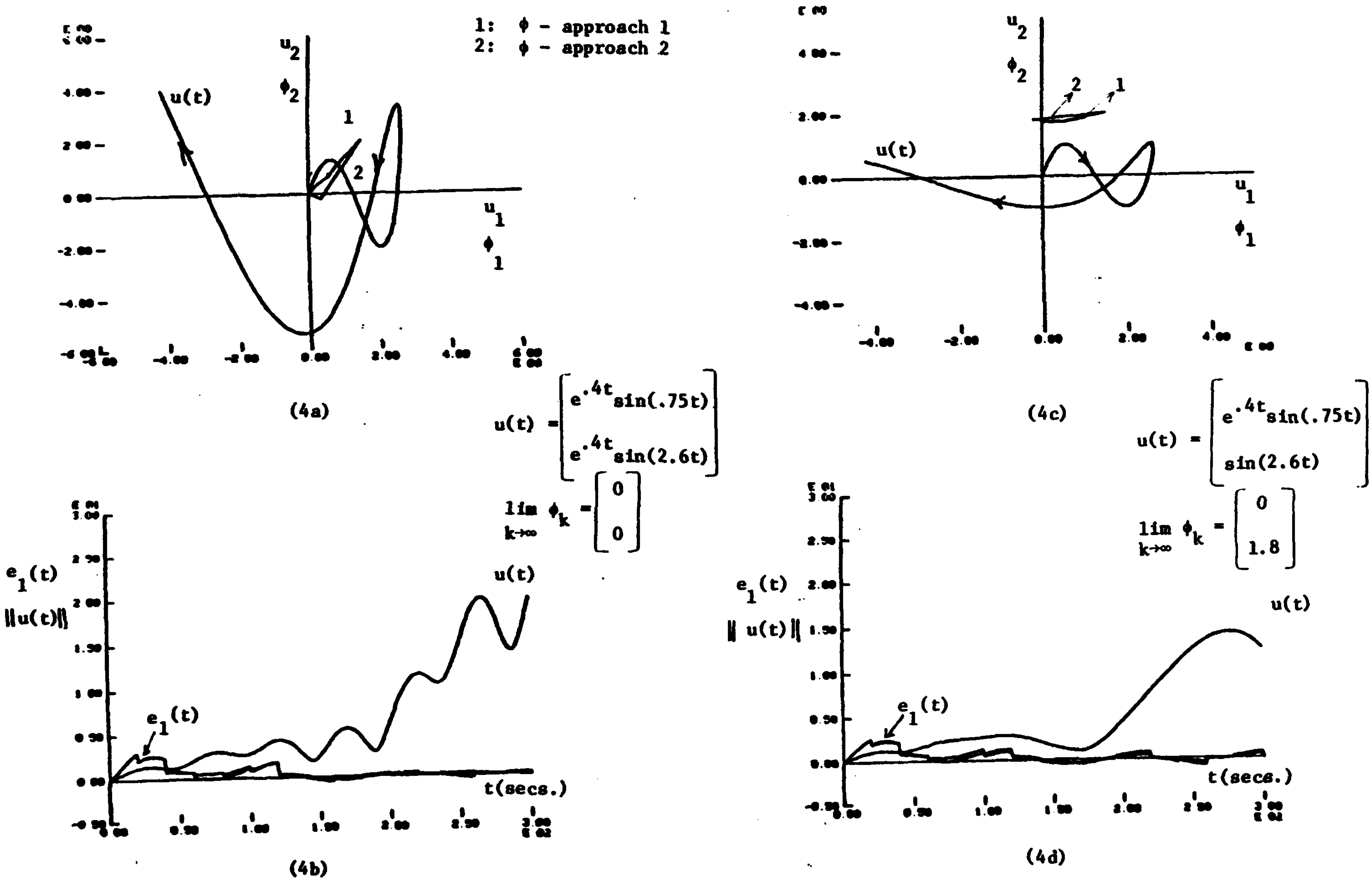

Figure 4: Response of error model 1 with exponentially growing input.

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